ON THE PROBLEM OF HORIZONTAL HYDRODYNAMIC IMPACT OF A SPHERE

(K ZADACHE O GORIZONTAL'NOM GIDRODINAMICHESKOM UDARE SPERY)

PMM Vol.22, No.6, 1958, pp.847-849

V.I. MOSSAKOVSKII and V.L. RVACHEV
(Dnepropetrovsk-Berdiansk)

(Received 29 October 1957)

The solution of the problem of the horizontal hydrodynamic impact of a sphere on a free fluid surface was found by Blokh [1] in the form of a series containing spherical (harmonic) functions. This note outlines a method by which a solution of this problem may be determined in closed form.

1. Let a spherical bowl be immersed in a fluid which fills the half-space $z \geqslant 0$, so that its wetted surface has the equation $r^2 = x^2 + y^2 + z^2 = 1$. (The assumption that the radius of the sphere is equal to unity obviously does not affect the generality of the argument.)

Now suppose that the sphere suddenly acquires a velocity U_0 along the axis Ox. Then, allowing for the fact that the velocity potential $\phi(x, y, z)$ of the perturbed fluid motion is a harmonic function within the fluid domain, connected with the impulsive pressure p_t by the relation $p_t = -\rho\phi$, where ρ is the density of the fluid, we arrive at the following conditions:

$$\varphi(x, y, 0) = 0 \quad \text{when} \quad x^2 + y^2 > 1 \tag{1.1}$$

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial r} = U_0 x \quad \text{when} \quad x^2 + y^2 + z^2 = 1 \quad (z > 0) \tag{1.2}$$

$$\operatorname{grad}\varphi \to 0 \quad \text{when} \quad x^2 + y^2 + z^2 \to \infty \tag{1.3}$$

It may easily be shown that the function

$$\psi(x, y, z) = r\frac{\partial\varphi}{\partial r} = x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z} \tag{1.4}$$

like $\phi(x, y, z)$, is a harmonic function. From the conditions (1.1) and (1.2) we find that

$$\psi(x, y, 0) = 0 \quad \text{when } x^2 + y^2 > 1 \tag{1.5}$$

$$\psi(x, y, z) = U_0 x \quad \text{when } x^2 + y^2 + z^2 = 1 \qquad (z > 0) \tag{1.6}$$

Let

$$\Psi(x, y, z) = \psi(x, y, z) - U_0 \frac{x}{r^3} \tag{1.7}$$

Then $\Psi(x, y, z) = 0$ when $r = 1$. It follows from Kelvin's theorem that the function

$$\Psi^*(x, y, z) = -\frac{1}{r} \Psi\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{z^2}\right)$$

will be harmonic in the domain $r < 1$. It is obvious that $\Psi = \Psi^* = 0$ when $r = 1$ and, moreover, we have

$$\left.\frac{\partial \Psi^*}{\partial r}\right|_{r=1} = \left.\left(\frac{1}{r^2} \Psi + \frac{1}{r^3} \frac{\partial \Psi}{\partial r}\right)\right|_{r=1} = \left.\frac{\partial \Psi}{\partial r}\right|_{r=1} \tag{1.8}$$

Thus the function $\Psi^*(x, y, z)$ proves to be the analytical continuation of the function $\Psi(x, y, z)$ across the sphere $r = 1$.

Let $F(x, y, z)$ be the function which equals $\Psi(x, y, z)$ when $r > 1$ and $\Psi^*(x, y, z)$ when $r < 1$. Then $F(x, y, z)$ will be a harmonic function in the half-space $z > 0$, satisfying the following boundary conditions on $z = 0$:

$$F(x, y, 0) = \begin{cases} U_0 x & \text{when } x^2 + y^2 < 1 \\ -\dfrac{U_0 x}{(x^2 + y^2)^{3/2}} & \text{when } x^2 + y^2 > 1 \end{cases} \tag{1.9}$$

2. Let $F(x, y, z) = F_1(x, y, z) + F_2(x, y, z)$, where F_1 and F_2 are harmonic functions satisfying the boundary conditions

$$F_1(x, y, 0) = \begin{cases} U_0 x & \text{when } x^2 + y^2 < 1 \\ 0 & \text{when } x^2 + y^2 > 1 \end{cases} \tag{2.1}$$

$$F_2(x, y, 0) = \begin{cases} 0 & \text{when } x^2 + y^2 < 1 \\ -\dfrac{U_0 x}{(x^2 + y^2)^{3/2}} & \text{when } x^2 + y^2 > 1 \end{cases} \tag{2.2}$$

It may easily be shown that

$$F_2(x, y, z) = -\frac{1}{r} F_1\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$$

so that

$$F(x, y, z) = F_1(x, y, z) - \frac{1}{r} F_1\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right) \qquad (2.3)$$

We determine the function $F_1(x, y, z)$ by solving the Dirichlet problem for the half-space:

$$F_1(x, y, z) = -\frac{U_0 z}{2\pi} \iint\limits_{\xi^2+\eta^2 \leqslant 1} \frac{\xi \, d\xi \, d\eta}{\left(\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}\right)^3} \qquad (2.4)$$

Using the formulas (1.4), (1.7), (2.3) and transforming to spherical coordinates

$$x = r \sin\theta \cos\omega, \quad y = r \sin\theta \sin\omega, \quad z = r \cos\theta$$

we get

$$\varphi(r, \theta, \omega) = -\frac{U_0 \sin\theta \cos\omega}{2r^2} - \frac{U_0}{2\pi} \cos\theta \int_0^{2\pi} \cos\alpha \, d\alpha \left\{ \int_r^\infty dr \int_0^{1/r} \frac{t^2 \, dt}{[1 - 2t \sin\theta \cos(\omega - \alpha) + t^2]^{3/2}} - \right.$$

$$\left. - \int_r^\infty \frac{dr}{r^3} \int_0^r \frac{t^2 \, dt}{[1 - 2t \sin\theta \cos(\omega - \alpha) + t^2]^{3/2}} \right\} \qquad (2.5)$$

In particular, it can be shown that when $r = 1$,

$$\varphi(1, \theta, \omega) = -\frac{U_0}{2} \sin\theta \cos\omega - \frac{U_0}{2\pi} \cos\theta \cos\omega \int_0^{2\pi} P(\sin\theta \cos\alpha) \cos\alpha \, d\alpha \qquad (2.6)$$

where

$$P(z) = \frac{1 + 3z}{(1 + z)\sqrt{2(1 - z)}} + \frac{1 - 3z}{2(1 - z^2)} - \frac{3}{2} \ln\left(1 + \frac{\sqrt{2}}{\sqrt{1 - z}}\right) \qquad (2.7)$$

Similarly, it is possible to obtain the solution of the so called "internal" problem of the horizontal impact of a spherical bowl half filled with fluid. In this problem, for instance,

$$\varphi^*(1, \theta, \omega) = U_0 \sin\theta \cos\omega - \frac{U_0}{2} \cos\theta \cos\omega \int_0^{2\pi} P^*(\sin\theta \cos\alpha) \cos\alpha \, d\alpha \qquad (2.8)$$

where

$$P^*(z) = -\frac{1 + 3z}{(1 + z)\sqrt{2(1 - z)}} + \frac{2z}{1 - z^2} + \frac{3}{2} \ln\left(1 + \frac{\sqrt{2}}{\sqrt{1 - z}}\right) \qquad (2.9)$$

3. For comparison with the solution given by Blokh, we calculate the virtual mass coefficient of the sphere λ_x, given by the formula

$$\lambda_x = -\frac{P_t}{{}^2/_3\, \rho\pi U_0} \tag{3.1}$$

where P_t is the resultant of the impulsive pressure forces acting on the wetted surface of the sphere:

$$P_t = -\iint_{(s)} p_t \cos(n, Ox)\, ds \qquad (ds = \sin\theta\, d\theta\, d\omega) \tag{3.2}$$

Remembering that $p_t = -\rho\phi$, we thus have

$$\lambda_x = \frac{1}{{}^2/_3 \rho\pi U_0} \int_0^{2\pi} \cos\omega\, d\omega \int_0^{{}^1/_2\pi} \sin^2\theta\varphi(1, \theta, \omega)\, d\theta \tag{3.3}$$

Hence, using the relation (2.6) and evaluating the integrals, we get

$$\lambda_x = \frac{4}{\pi} - 1 = 0.27323954 \tag{3.4}$$

Similarly for the internal problem, we find that

$$\lambda_x^* = 4\pi^{-1} - {}^7/_8 = 0.39823954$$

The following values of the same coefficients are given in the work of Blokh: $\lambda_x = 0.27322$, $\lambda_x^* = 0.39822$.

BIBLIOGRAPHY

1. Blokh, E.L., Gorizontal'nyi gidrodinamicheskii udar sfery pri nalichii svobodnoi poverkhnosti zhidkosti (The horizontal hydrodynamic impact of a sphere on a free fluid surface). *PMM* Vol. 17, No. 5, 1953.